VNiVERSiDAD
D SALAMANCA

Simple majority rule: a model with voice but no vote

José Carlos R. Alcantud

Catania, June 5th 2019

BORDA Research Unit and IME, Universidad de Salamanca, Spain

Target and distinctive features

Axiomatizations are useful for specifying the scope of a model.

Suppose that a voting rule F produces the following aggregate outcomes when the electorate is formed by 3 agents, and $-1, 0$ and 1 respectively mean rejection, unresolvedness and acceptance:

$F(1, 0, -1) = 1$, $F(1, -1, 0) = 0$, $F(0, 0, 1) = 0$, $F(0, 0, -1) = 0$.

The simple majority is anonymous. A comparison of the first and second aggregations of the opinions discards that F is the simple majority rule.

And the third and fourth aggregation results contradict another axiom in May's characterization, namely, positive responsiveness.

However these realizations of the voting rule are not fully incompatible with the spirit of relative majoritarianism:

▷ They can be explained by an *oligarchic majority rule* (i.e., a majority rule where only the votes of an oligarchy are counted), the oligarchy being formed by the first and second agents.

Target and distinctive features

Different applications of this rule are observed in practice.

▷ Puerto Rico has one Resident Commissioner in the U. S. House of Representatives who has a voice but no vote.

▷ Guam and the Virgin Islands each has one Delegate in the U. S. House of Representatives (since 1972 and 1973, respectively) with voice but no vote.

▷ The Security Council Rules of Procedure establish in Rule 5 "[...] *the Security Council may invite observers to attend its deliberations, [...] observers shall have a voice but no vote in matters both substantive and procedural*."

Rule 6 states "*UN Member States that are not members of the Security Council and States that are not members of the United Nations may take part in the Security Council through the figure of the Representative to the Council. The representatives appointed to this effect shall have a voice but no vote in matters both substantive and procedural*."

Target and distinctive features

The first objective of this work is to provide an axiomatic basis for oligarchic majority rules, both with a fixed and a variable population.

For each framework we identify sets of axioms that reveal the existence of an oligarchy whose opinions are aggregated by a simple majoritarian spirit.

Alternatively, the axioms identify a (possibly empty) subsociety whose members have voice but no vote, and valid votes are aggregated by the majority rule.

Table of contents

Preliminary concepts

Basic elements I

Let **A** be a subset of two alternatives, say x and y. We can also think of an issue that passes (alternative x) or is defeated (alternative y).

A *society* is a non-empty set $N = \{1, \ldots, n\}$ of *voters* or agents.

Voter i has complete and transitive preferences over **A**, $R_i \in \{-1, 0, 1\}$.

▷ $R_i = 1$, resp., $R_i = -1$, means i strictly prefers x to y (or i wants that the issue passes), resp., y to x (or i wants that the issue is defeated).

In either case we say that voter i is resolved about **A**.

▷ We write $R_i = 0$ when agent i is indifferent between x and y.

The society's preferences are collected in a preference profile $R = (R_1, \ldots, R_n) \in \{-1, 0, 1\}^n$ whose *length* is n.

Basic elements II

The collection of preference profiles for arbitrarily large non-empty societies is $\mathbf{P} = \bigcup_{n>0}\{-1, 0, 1\}^n$.

For each $R, R' \in \mathbf{P}$ with length n, we denote $R \geqslant R'$ when $R_i \geqslant R'_i$ for each $i = 1, \ldots, n$, and we write $R > R'$ when $R \geqslant R'$ but $R \neq R'$.

A social welfare function or SWF is a mapping $F : D \longrightarrow \{-1, 0, 1\}$, with $D \subseteq \bigcup_{n>0}\{-1, 0, 1\}^n$.

▷ $F(R) = 1$ means that the issue passes, $F(R) = -1$ means that it is defeated, and $F(R) = 0$ means that it is unresolved.

The majority rule is the SWF denoted F_M that assigns to each $R \in \mathbf{P}$ with length n the collective preference $F_M(R) = \text{sgn}(\sum_{i \in N} R_i)$. Here sgn denotes the usual sign function on the real numbers, i.e., $\text{sgn}(x)$ equals $1, 0, -1$ when $x > 0, x = 0, x < 0$, respectively.

Its axiomatization by May (Econometrica, 1952) anticipated the importance of the Arrovian framework for the analysis of voting rules.

Other characterizations of the majority rule

Aşan and Sanver, Economics Letters, 2002.

Alcantud, Economics Letters, 2019.

Campbell and Kelly, Economics Theory, 2000.

Fishburn, The Theory of Social Choice, 1973.

Fishburn, Economics Letters, 1983.

Llamazares, Mathematical Social Sciences, 2006.

Miroiu, Economics Letters, 2004.

Quesada, Mathematical Social Sciences, 2010.

Quesada, Economics Bulletin, 2010.

Woeginger, Economics Letters, 2003.

Xu and Zhong, Economics Letters, 2010.

Yi, Economics Letters, 2005.

et cetera.

Summary of characterizations of the majority rule

Summary of existing characterizations of the majority rule in the context of this paper. The list is not exhaustive.

	A	N	PR	PO	RS	APR	WPI	LR	C
May 1952	✓	✓	✓						
Fishburn 1973		✓	✓					✓	
Așan and Sanver 2002	✓	✓		✓			✓		
Woeginger 2003		✓		✓	✓				
Woeginger 2005	✓	✓				✓			
Llamazares 2006		✓		✓					✓

May, Fishburn and Llamazares consider a fixed society.

Fishburn uses a variation of PR that is equivalent to a weak specification of PR under N.

There are other characterizations, like Fishburn (1983), who considers a society with a fixed number of agents, or Quesada (2010a,2010b).

Further characterizations of the majority rule

We can also cite other contexts where the majority rule has been investigated. For example:

▷ Yi (2005) studies majority and weak majority rules for fixed societies and arbitrary agendas.

▷ Campbell and Kelly (2013) consider a binary agenda and a social choice function that cannot declare a tie between the two options.

▷ Dasgupta and Maskin (2008) assume a continuum of voters who can never be indifferent between two alternatives.

▷ Quesada (2013) complements the majority rule with a ranking among individuals such that in case of social indifference, the non-indifferent agent that ranks first determines the social preference.

▷ Xu and Zhong (2010) refer to a set of individuals that is variable but whose preferences remain fixed.

Model and axiomatization with a fixed population

Axiomatization with a fixed population

We fix a population size n. Then we are interested in the following model:

Definition

A social welfare function $F : \{-1, 0, 1\}^n \longrightarrow \{-1, 0, 1\}$ is an oligarchic majority rule if there is $\varnothing \neq A \subseteq N$ such that $F = F^A$, where for each $R \in \{-1, 0, 1\}^n$, $F^A(R) = \text{sgn}(\sum_{i \in A} R_i)$.

- If $A = \{i\}$ for some i we say that F^A is dictatorial majoritarian.
- If $A = N$ then F^A is the simple majority rule.

We prove a characterization of the SWFs that verify this Definition.

Our technique produces two further characterizations of the simple majority rule.

Axioms

The following two axioms are standard in the analysis of SWFs:

Neutrality (N). For any $R \in \{-1, 0, 1\}^n$, $F(R) = -F(-R)$.

Monotonicity (MON). For any $R, R' \in \{-1, 0, 1\}^n$ such that $R \geqslant R'$, $F(R) \geqslant F(R')$.

We use further technical notation.

We denote by R^i the profile such that $R_i = 1$ and $R_j = 0$ when $j \neq i$.

With every $R \in \{-1, 0, 1\}^n$ and $i \in N$ we associate the profile $\hat{R}^i \in \{-1, 0, 1\}^n$ such that $\hat{R}^i_j = 0$ when $j \neq i$, $\hat{R}^i_i = R_i$.

Reduction to individuals (RI). For any $R \in \{-1, 0, 1\}^n$,

$$F(R) = F(F(\hat{R}^1), \ldots, F(\hat{R}^n)).$$

Technical Lemma I

We say that $R \in \{-1, 0, 1\}^n$ is irrelevant-free (IF) with respect to F when it verifies $R_i = F(\hat{R}^i)$ for each $i \in N$, or equivalently, $R = (F(\hat{R}^1), \ldots, F(\hat{R}^n))$.

With this notion we refer to profiles where the vote of each individual coincides with the result of the vote when she is the only voter that is resolved about **A** and she casts her vote.

Note that RI holds true when all the profiles are IF. In particular, this is the case when the SWF is the simple majority rule (i.e., the simple majority rule verifies RI because every profile is IF for it).

Lemma

Let us fix $F : \{-1, 0, 1\}^n \longrightarrow \{-1, 0, 1\}$, a SWF that is N and MON. Then $R \in \{-1, 0, 1\}^n$ is IF if and only if $F(\hat{R}^i) = 0$ implies $R_i = 0$ whenever $i \in N$.

Technical Lemma II

With the next technical result we identify a set of properties for which any profile can be associated with an IF profile that produces the same social output, in a constructive fashion:

Lemma

Let us fix $F : \{-1, 0, 1\}^n \longrightarrow \{-1, 0, 1\}$, a SWF that is RI, N and MON. For every $R \in \mathbf{P}$ with length n, let $A_R = \{i \in N : F(\hat{R}^i) \neq 0\}$.

Let P be the profile defined as $P_i = R_i$ when $i \in A_R$, $P_i = 0$ otherwise.

Then $F(R) = F(P)$ and P is IF.

Therefore N, MON and RI are the properties of a SWF that reveal the existence of an oligarchy with some privileges. When a member of the oligarchy is the only resolved voter, then her position on the issue is accepted as the social aggregate. However under any circumstances, the position of an agent not belonging to the oligarchy is disregarded (the social output does not change if her vote is replaced by indifference).

Axioms (continued)

We need another new axiom. It is a generalization of the standard cancellation property that has been used to characterize M_k-majorities (Llamazares, MaSS 2006) like the simple majority rule, or the Borda rule (Young, JET 1974):

Cancellation (C). For any $R, R' \in \{-1, 0, 1\}^n$ such that there are $i, j \in \{1, \ldots, n\}$ with $R'_k = R_k$ for $i \neq k \neq j$, $R_i = 1 = -R_j$, $R'_i = 0 = R'_j$, it must be the case that $F(R) = F(R')$.

C means that the aggregate of a profile does not change when two opposing opinions are replaced by two indifferences.

IF-Cancellation (IF-C). For any $R, R' \in \{-1, 0, 1\}^n$ such that R is IF, and there are $i, j \in \{1, \ldots, n\}$ with $R'_k = R_k$ for $i \neq k \neq j$, $R_i = 1 = -R_j$, $R'_i = 0 = R'_j$, it must be the case that $F(R) = F(R')$.

IF-C only requests that the aggregate of a profile that is IF does not change if two opposing opinions are replaced by two indifferences.

Characterization with a fixed population

We are ready to state our characterization of oligarchic majority rules in a society with fixed population:

Theorem

A social welfare function $F : \{-1, 0, 1\}^n \longrightarrow \{-1, 0, 1\}$ verifies N, MON, RI and IF-C if and only if it is an oligarchic majority rule.

The set of axioms in this Theorem is tight:

Proposition

There are social welfare functions F_1, F_2, F_3, F_4, other than oligarchic majority rules, such that

(a) F_1 verifies N, MON and IF-C, but not RI.

(b) F_2 verifies RI, MON and IF-C but not N.

(c) F_3 verifies RI, N and IF-C but not MON.

(d) F_4 verifies RI, N and MON, but not IF-C.

Characterizations of simple majority with a fixed population

We derive an original characterization of the simple majority rule that uses the following property from May's characterization of this rule:

Positive responsiveness (PR). For any $R, R' \in \mathbf{P}$ with length n and $R > R'$, (a) $F(R') \geqslant 0$ implies $F(R) = 1$, and (b) $F(R) \leqslant 0$ implies $F(R') = -1$.

Corollary

A social welfare function $F : \{-1, 0, 1\}^n \longrightarrow \{-1, 0, 1\}$ verifies N, PR and C if and only if it is the simple majority rule.

There are SWFs other than the majority rule, for which every profile is IF. However it is the only SWF that in addition, verifies MON and C.

Proposition

A social welfare function $F : \{-1, 0, 1\}^n \longrightarrow \{-1, 0, 1\}$ verifies MON, C and every profile is IF for F, if and only if it is the simple majority rule.

Model and axiomatization with a variable population

Axiomatization with a variable population

Definition

A social welfare function $F : \bigcup_{n>0}\{-1,0,1\}^n \longrightarrow \{-1,0,1\}$ is an oligarchic majority rule if there is $\varnothing \neq A \subseteq \mathbb{N}$ such that $F = F^A$, where for each $R \in \{-1,0,1\}^n$,

$$F^A(R) = \begin{cases} \operatorname{sgn}(\sum_{i\in A\cap N} R_i), & \text{if } A \cap \{1.\dots,n\} \neq \varnothing, \\ 0, & \text{if } A \cap \{1.\dots,n\} = \varnothing. \end{cases}$$

- When $A = \{i\}$ for some i we say that F^A is dictatorial majoritarian.
- If $A = \mathbb{N}$ then F^A is the simple majority rule.

Our axioms for SWFs heavily rely on agents with the feature:
agent i is relevant for F if $F(0, \overset{i-1}{\dots}, 0, 1) \neq 0$.
Otherwise the agent is irrelevant for F.

When F^A is the oligarchic majority rule associated with $A \subseteq \mathbb{N}$, A coincides with the set of relevant agents of F^A.

Axioms I

R-Additive responsiveness (R-AR). For any $R \in \mathbf{P}$ with length n and $F(R) \geqslant 0$, resp., $F(R) \leqslant 0$, it must be the case that if $n+1$ is relevant, then $F(R_1, \ldots, R_n, 1) = 1$, resp., $F(R_1, \ldots, R_n, -1) = -1$.

R-AR is weaker than Additive responsiveness, defined by Miroiu (Economics Letters, 2004) in his characterization of the simple majority rule. It focuses on the behavior of the rule under expansions of the society that incorporate a new relevant agent.

We can insist on expansions that add up irrelevant agents by the following property:

I-Responsiveness (I-R). For any $n \in \mathbb{N}$ and $R \in \mathbf{P}$ with length n, if $n+1$ is irrelevant then $F(R) = F(R, R_{n+1})$ for each $R_{n+1} \in \{-1, 0, 1\}$.

I-R assures that the addition of irrelevant agents produces the same output as the rule did without the agent.

Axioms II

R-Neutrality (R-N). For any $n \in \mathbb{N}$ and $R, R' \in \mathbf{P}$ with length n such that $R'_i = -R_i$ for each relevant $i \in N$, $F(R) = -F(R')$.

R-Neutrality claims that if every relevant agent reverses her preference between x and y, then the social preference is reversed too.

It implies the standard Neutrality (N) axiom, which states $F(-R) = -F(R)$ for any $n \in \mathbb{N}$ and $R \in \mathbf{P}$. Thus $F(0) = 0$ under R-N.

If F verifies R-N and 1 is irrelevant, then $F(R_1) = -F(R_1)$ for each $R_1 \in \{-1, 0, 1\}$ therefore $F(R_1) = 0$ for each $R_1 \in \{-1, 0, 1\}$.

Technical Lemmas

Lemma

If $F : \mathbf{P} \longrightarrow \{-1, 0, 1\}$ is an R-Neutral social welfare function and i is irrelevant,

$$F(R_1, \ldots, R_{i-1}, 1, R_{i+1}, \ldots, R_n) = F(R_1, \ldots, R_{i-1}, -1, R_{i+1}, \ldots, R_n) = \\ = F(R_1, \ldots, R_{i-1}, 0, R_{i+1}, \ldots, R_n)$$

irrespective of the opinions $R_1, \ldots, R_{i-1}, R_{i+1}, \ldots, R_n$.

▷ Under R-N, the opinions of irrelevant agents are irrelevant.

Lemma

If $F : \mathbf{P} \longrightarrow \{-1, 0, 1\}$ is an R-Neutral social welfare function and $R, R' \in \{-1, 0, 1\}^n$ verify $R_i = R'_i$ for each relevant $i \in N$, then $F(R) = F(R')$.

▷ Under R-N, if two profiles differ in the opinions of irrelevant agents only, then they produce the same social output.

Axioms (continued)

Instead of the standard Anonymity axiom we only require the following weakened version:

R-Anonymity (R-A). For any $n \in \mathbb{N}$, $R \in \mathbf{P}$ with length n, and any permutation Π of N such that $\Pi(i)$ is relevant whenever i is relevant, one has $F(R) = F(R_{\Pi(1)}, \ldots, R_{\Pi(n)})$.

Finally, in order to distinguish dictatorships from other behaviors we use property 1R (there is exactly one relevant agent), whose negation is denoted by N1R.

Characterization with a variable population

Theorem

A social welfare function $F : \mathbf{P} \longrightarrow \{-1, 0, 1\}$ verifies R-AR, I-R, R-N, R-A, and N1R if and only if it is an oligarchic majority rule which is not a dictatorship.

The axiomatics in this characterization is tight:

Proposition

There are social welfare functions F_1, F_2, F_3, F_4, F_5, other than oligarchic majority rules, such that

(a) F_1 verifies R-AR, I-R, R-N, and R-A, but it verifies 1R.

(b) F_2 verifies I-R, R-N, R-A, and N1R, but not R-AR.

(c) F_3 verifies R-AR, R-N, R-A, and N1R, but not I-R.

(d) F_4 verifies R-AR, I-R, R-A, and N1R, but not R-N.

(e) F_5 verifies R-AR, I-R, R-N, and N1R, but not R-A.

Characterization of simple majority with a variable population

We derive the characterization of the majority rule in Woeginger (Economics Letters, 2005), Theorem 3, which improved a previous characterization in Miroiu (Economics Letters, 2004), Theorem 2.

We only need to observe that when every agent is relevant I-R is vacuously verified, N1R is obvious, R-A is the standard Anonymity axiom, and R-N is the standard Neutrality axiom; and in addition, R-AR is Additive positive responsiveness (APR) in these two references.

From this observation one readily derives:

Corollary (Woeginger, 2005)

A social welfare function $F : \mathbf{P} \longrightarrow \{-1, 0, 1\}$ verifies N, A and APR if and only if it is the simple majority rule.

Conclusion

Summary

Oligarchic majority rules are the formal implementation of the *voice but no vote* principle. Here we have studied the case of two-alternative elections where valid votes are aggregated by the majority rule. Both the oligarchy and the voting rule are derived endogenously from our setup.

For a given population, axioms Monotonicity and Cancellation are compatible with many social welfare functions. Among them however, only the simple majority verifies that every profile is irrelevant-free. An irrelevant-free profile coincides with the profile where every voter behaves as the voting outcome when the only resolved vote is hers.

When not every profile is irrelevant-free, Neutrality together with Reduction to individuals results into an oligarchic majority rule. Reduction to individuals means that when a profile is replaced with the profile where every voter behaves as the SWF prescribes when the only resolved vote is hers, the aggregate output does not change.

Summary

When the population is not fixed but is a finite part of a potentially infinite society, then we provide axioms that characterize the oligarchic majority rules by the recourse to properties that focus on the role of relevant agents.
They are identified by a very simple characteristic, and they ultimately coincide with the oligarchy of voters with voice and a vote.

Despite the negative antecedents in the literature, it is possible to characterize the majority rule with neither of the axioms in May's original characterization.

Thank you!

References i

- Alcantud, J. C. R., 2019. Yet another characterization of the majority rule. Economics Letters 177, 52-55.
- Aşan, G., Sanver, R., 2002. Another characterization of the majority rule. Economics Letters 95, 409-413.
- Campbell, D.E., Kelly, J.S., 2000. A simple characterization of majority rule. Economic Theory 15, 689-700.
- Campbell, D.E., Kelly, J.S., 2013. Anonymity, monotonicity, and limited neutrality: Selecting a single alternative from a binary agenda. Economics Letters 118, 10-12.
- Dasgupta, P.S., Maskin, E.S., 2008. On the robustness of majority rule. Journal of the European Economic Association 6, 949-973.
- Fishburn, P.C., 1983. A new characterization of simple majority. Economics Letters 13, 31-35.

References ii

- Llamazares, B., 2006. The forgotten decision rules: Majority rules based on difference of votes. Mathematical Social Sciences 51, 311-326.
- May, K., 1952. A set of independent, necessary and sufficient conditions for simple majority decision. Econometrica 20, 680-684.
- Miroiu, A., 2004. Characterizing majority rule: from profiles to societies. Economics Letters 85, 359-363.
- Ozkes, A.I., Sanver, M.R., 2017. Absolute qualified majoritarianism: How does the threshold matter? Economics Letters 153, 20-22.
- Quesada, A., 2010a. Monotonicity + efficiency + continuity = majority. Mathematical Social Sciences 60, 149-153.
- Quesada, A., 2010b. Two axioms for the majority rule. Economics Bulletin 30(4), 3033-3037.

- Quesada, A., 2013. The majority rule with arbitrators. Group Decision and Negotiation 22, 321-330.
- Woeginger, G.J., 2003. A new characterization of the majority rule. Economics Letters 81, 89-94.
- Xu, Y., Zhong, Z. (2010) Single profile of preferences with variable societies: a characterization of simple majority rule. Economics Letters 107, 119-121.
- Yi, J., 2005. A complete characterization of majority rules. Economics Letters 87, 109-112.