The simple majority rule: old and new axiomatics

José Carlos R. Alcantud

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BORDA Research Unit and IME, Universidad de Salamanca, Spain

Target

Axiomatizations are useful for specifying the scope of a model.

The first objective of this work is to recall the extensive axiomatic basis for the majority rule, both with a fixed and a variable population.

Then we present a characterization of the simple majority rule for unrestricted societies.

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Preliminary concepts

Basic elements I

Let **A** be a subset of two alternatives, say x and y. We can also think of an issue that passes (alternative x) or is defeated (alternative y).

A *society* is a non-empty set $N = \{1, \ldots, n\}$ of *voters* or agents.

Voter i has complete (and transitive) preferences over **A**, $R_i \in \{-1, 0, 1\}$.

▷ $R_i = 1$, resp., $R_i = -1$, means i strictly prefers x to y (or i wants that the issue passes), resp., y to x (or i wants that the issue is defeated).

In either case we say that voter i is resolved about **A**.

▷ We write $R_i = 0$ when agent i is indifferent between x and y.

The society's preferences are collected in a preference profile $R = (R_1, \ldots, R_n) \in \{-1, 0, 1\}^n$ whose *length* is n.

Basic elements II

The collection of preference profiles for arbitrarily large non-empty societies is $\mathbf{P} = \bigcup_{n>0}\{-1,0,1\}^n$.

For each $R, R' \in \mathbf{P}$ with length n, we denote $R \geqslant R'$ when $R_i \geqslant R'_i$ for each $i = 1, \ldots, n$, and we write $R > R'$ when $R \geqslant R'$ but $R \neq R'$.

A social welfare function or SWF is a mapping $F : D \longrightarrow \{-1,0,1\}$, with $D \subseteq \bigcup_{n>0}\{-1,0,1\}^n$.

▷ $F(R) = 1$ means that the issue passes, $F(R) = -1$ means that it is defeated, and $F(R) = 0$ means that it is unresolved.

The majority rule is the SWF denoted F_M that assigns to each $R \in D$ with length n the collective preference $F_M(R) = \mathrm{sgn}(\sum_{i \in N} R_i)$. Here sgn denotes the usual sign function on the real numbers, i.e., $\mathrm{sgn}(x)$ equals $1, 0, -1$ when $x > 0, x = 0, x < 0$, respectively.

Its axiomatization by May (Econometrica, 1952) anticipated the importance of the Arrovian framework for the analysis of voting rules.

Other characterizations of the majority rule

Așan and Sanver, Economics Letters, 2002.

Alcantud, Economics Letters, 2019. Subject of this talk.

Campbell and Kelly, Economics Theory, 2000.

Fishburn, The Theory of Social Choice, 1973.

Fishburn, Economics Letters, 1983.

Llamazares, Mathematical Social Sciences, 2006.

Miroiu, Economics Letters, 2004.

Quesada, Mathematical Social Sciences, 2010.

Quesada, Economics Bulletin, 2010.

Woeginger, Economics Letters, 2003.

Xu and Zhong, Economics Letters, 2010.

Yi, Economics Letters, 2005.

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Summary of characterizations of the majority rule

Summary of existing characterizations of the majority rule. The list is not exhaustive.

	A	N	PR	PO	RS	APR	WPI	LR	C
May 1952	✓	✓	✓						
Fishburn 1973		✓	✓					✓	
Aşan and Sanver 2002	✓	✓		✓			✓		
Woeginger 2003		✓		✓	✓				
Woeginger 2005	✓	✓				✓			
Llamazares 2006		✓		✓					✓

May, Fishburn and Llamazares consider a fixed society.

Fishburn uses a variation of PR that is equivalent to a weak specification of PR under N.

There are other characterizations, like Fishburn (1983), who considers a society with a fixed number of agents, or Quesada (2010a,2010b).

Further analyses of the majority rule

We can also cite other contexts where the majority rule has been investigated. For example:

▷ Yi (2005) studies majority and weak majority rules for fixed societies and arbitrary agendas.

▷ Campbell and Kelly (2013) consider a binary agenda and a social choice function that cannot declare a tie between the two options.

▷ Dasgupta and Maskin (2008) assume a continuum of voters who can never be indifferent between two alternatives.

▷ Quesada (2013) complements the majority rule with a ranking among individuals such that in case of social indifference, the non-indifferent agent that ranks first determines the social preference.

▷ Xu and Zhong (2010) refer to a set of individuals that is variable but whose preferences remain fixed.

Model and axiomatizations with a fixed population

The majority rule with a fixed population

We fix a population size n. Then we are interested in the following model:

Definition

The majority rule is the social welfare function $F_M : \{-1, 0, 1\}^n \longrightarrow \{-1, 0, 1\}$ that assigns to each $R \in \{-1, 0, 1\}^n$ the collective preference $F_M(R) = \text{sgn}(\sum_{i=1}^n R_i)$.

Axioms

The first axiom is standard in the analysis of SWFs:

Neutrality (N). For any $R \in \{-1, 0, 1\}^n$, $F(R) = -F(-R)$.

Anonymity is standard too, and it requests:

Anonymity (A). For any $n \in \mathbb{N}$, $R \in \mathbf{P}$ with length n, and any permutation Π of N, one has $F(R) = F(R_{\Pi(1)}, \ldots, R_{\Pi(n)})$.

May's theorem

Some characterizations of the simple majority rule use the following property from May's pioneering analysis:

Positive responsiveness (PR). For any $R, R' \in \mathbf{P}$ with length n and $R > R'$,

(a) $F(R') \geqslant 0$ implies $F(R) = 1$, and

(b) $F(R) \leqslant 0$ implies $F(R') = -1$.

Theorem (May, 1952)

A social welfare function $F : \{-1, 0, 1\}^n \longrightarrow \{-1, 0, 1\}$ is the simple majority rule if and only if it verifies N, A and PR.

Monotonicity-type axioms

Variations of Pareto optimality are a must in this context. For example:

Monotonicity (MON). For any $R, R' \in \{-1, 0, 1\}^n$ such that $R \geqslant R'$, $F(R) \geqslant F(R')$.

Pareto optimality (PO). For any $R \in \mathbf{P}$ with length n such that $R \neq \mathbf{0}_n$, (a) if $R \geqslant \mathbf{0}$ then $F(R) = 1$, and (b) if $\mathbf{0} \geqslant R$ then $F(R) = -1$.

Fishburn uses a limited responsiveness notion:

Limited responsiveness (LR). For any $R, R' \in \mathbf{P}$ with length n, if $F(R) = 1$ and there is $j \in N$ such that $R_i - R'_i = 0$ for all $j \neq i \in N$ and $R_j - R'_j = 1$, then one has $F(R') \geqslant 0$.

Two characterizations with a fixed population

The following cancellation property has been used to characterize the Borda rule (Young, JET 1974) and M_k-majorities (Llamazares, MaSS 2006):

Cancellation (C). For any $R, R' \in \{-1, 0, 1\}^n$ such that there are $i, j \in \{1, \ldots, n\}$ with $R'_k = R_k$ for $i \neq k \neq j$, $R_i = 1 = -R_j$, $R'_i = 0 = R'_j$, it must be the case that $F(R) = F(R')$.

C means that the aggregate of a profile does not change when two opposing opinions are replaced by two indifferences.

Theorem (Fishburn, 1973)

A social welfare function $F : \{-1, 0, 1\}^n \longrightarrow \{-1, 0, 1\}$ is the simple majority rule if and only if it verifies N, PR and LR.

Theorem (Llamazares, 2006)

A social welfare function $F : \{-1, 0, 1\}^n \longrightarrow \{-1, 0, 1\}$ is the simple majority rule if and only if it verifies N, PO and C.

Model and axiomatizations with a variable population

The majority rule with a variable population

Now we are interested in the following model:

Definition

The majority rule is the social welfare function $F_M : \mathbf{P} \longrightarrow \{-1, 0, 1\}$ that assigns to each $R \in \{-1, 0, 1\}^n \subseteq \mathbf{P}$ the collective preference $F_M(R) = \text{sgn}(\sum_{i=1}^{n} R_i)$.

Axiomatization by Aşan and Sanver

Among the three properties in May's characterization, PR has been especially criticized for being too strong. Aşan and Sanver (2002) show that PR can be replaced by PO and the following property:

Weak path independence (WPI). For any $R, R' \in \mathbf{P}$ with respective lengths n and n' and such that $|F(R) - F(R')| \neq 2$, we have $F(R_1, \ldots, R_n, R'_1, \ldots, R'_{n'}) = F(F(R), F(R'))$.

The idea behind WPI goes as follows. Suppose two disjoint societies that are combined into a new society. Then in order to compute the social preference of the enlarged society, *provided that the two initial societies are not in total disagreement* you can instead aggregate their two preference profiles into their respective social representatives and then aggregate them.

Theorem (Aşan and Sanver, 2002)

A social welfare function $F : \mathbf{P} \longrightarrow \{-1, 0, 1\}$ is the simple majority rule if and only if it verifies A, N, PO and WPI.

Axioms I

Now the population is not fixed. Therefore we can think of the behavior of the rule when people are added or removed from a population.

Additive responsiveness (AR). For any $R \in \mathbf{P}$ with length n and $F(R) \geqslant 0$, resp., $F(R) \leqslant 0$, it must be the case that $F(R_1, \ldots, R_n, 1) = 1$, resp., $F(R_1, \ldots, R_n, -1) = -1$.

It was defined by Miroiu (Economics Letters, 2004) and focuses on the behavior of the rule under expansions of the society that incorporate a resolved agent.

When the aggregate preference is indifferent, and a resolved voter joins the society, then the new aggregate preference must follow this voter's opinion.

Characterization of simple majority with a variable population

The characterization of the majority rule in Woeginger (Economics Letters, 2005), Theorem 3, improved a previous characterization in Miroiu (Economics Letters, 2004), Theorem 2:

Theorem (Woeginger, 2005)

A social welfare function $F : \mathbf{P} \longrightarrow \{-1, 0, 1\}$ is the simple majority rule if and only if it verifies N, A and AR.

Another characterization of the majority rule for unrestricted societies

Yet another characterization of the majority rule

We continue in the setting of the last section (a variable population of voters, with complete preferences on a binary agenda).

We now present three axioms that provide a new characterization of the majority rule that uses neither of the axioms in May's original characterization.

We use the notation R^{-i} to refer to the profile derived from R by removing the opinion of agent i.
Related notations like $R^{-i,-j}$ should have obvious meanings.

Axioms

Individual consistency (IC). For any profile $R = R_1$ of length 1, $F(R_1) = R_1$.

Woeginger (2003) and Quesada (2010, MaSS) refer to IC. The collective preference of a society with only one member is his/her preference.

Non-swap (NS). For every $R \in \mathbf{P}$ with length $n > 1$, if $R_i \neq 0$ with $i \in N$, then $R_i \neq F(R)$ implies $F(R^{-i}) = -R_i$.

Non-swap bears some similarity to non-reversal in Campbell and Kelly (2006), the key property of the majority rule that results in an induced strategy-proof social choice rule (Campbell and Kelly, 2010).
It establishes that for voters that are resolved about $\mathbf{A}$, there are no incentives to manipulate the voting result by not showing up.

Social fairness (SF). For every $R \in \mathbf{P}$ with length $n > 1$, if $F(R^{-k}) = -R_k$ for all $k \in N$ then there are $i < j, i, j \in N$, such that $F(R^{-i,-j}) = F(R)$ and $F(R_i, R_j) = 0$.

A new characterization of the majority rule

Theorem

A social welfare function $F : \mathbf{P} \longrightarrow \{-1, 0, 1\}$ verifies IC, NS and SF if and only if it is the simple majority rule.

The set of axioms in this characterization is tight:

Proposition

There are social welfare functions F_1, F_2, F_3 other than F_M, such that

(a) F_1 verifies IC, SF but not NS.

(b) F_2 verifies NS, SF but not IC.

(c) F_3 verifies IC, NS but not SF.

The counterexamples

- ▷ F_1 that verifies IC, SF but not NS: for each $R \in \mathbf{P}$, $F_1(R) = 1$ if $R > \mathbf{0}$, $F_1(R) = -1$ if $\mathbf{0} > R$, and $F_1(R) = 0$ otherwise.

- ▷ F_2 that verifies NS, SF but not IC. We use the following notation: for each $R \in \mathbf{P}$, $\tilde{R}$ is the vector that results from replacing each 0 by a 1 in R, i.e., R and $\tilde{R}$ have equal length n, and for each $i = 1, \ldots, n$, $\tilde{R}_i = R_i$ if $R_i \neq 0$, $\tilde{R}_i = 1$ if $R_i = 0$. Observe that $\tilde{R}^{-i} = \widetilde{R^{-i}}$, $\tilde{R}^{-i,-j} = \widetilde{R^{-i,-j}}$ throughout.

  Now for each $R \in \mathbf{P}$, we let $F_2(R) = F_M(\tilde{R})$. Because $F_2(0) = 1$, F_2 contradicts IC.

- ▷ F_3 that verifies IC, NS but not SF: for each $R \in \mathbf{P}$,

$$F_3(R) = \begin{cases} F_M(R), & \text{if either } n = 1 \text{ or } (n > 1 \text{ and } F_M(R) \neq 0), \\ 1, & \text{if } n > 1 \text{ and } F_M(R) = 0 \end{cases}$$

A final comment

Our next seminar produces two further characterizations of the simple majority rule when the population is fixed.

Thank you!

- Alcantud, J. C. R., 2019. Yet another characterization of the majority rule. Economics Letters 177, 52-55.
- Aşan, G., Sanver, R., 2002. Another characterization of the majority rule. Economics Letters 95, 409-413.
- Campbell, D.E., Kelly, J.S., 2000. A simple characterization of majority rule. Economic Theory 15, 689-700.
- Campbell, D.E., Kelly, J.S., 2013. Anonymity, monotonicity, and limited neutrality: Selecting a single alternative from a binary agenda. Economics Letters 118, 10-12.
- Dasgupta, P.S., Maskin, E.S., 2008. On the robustness of majority rule. Journal of the European Economic Association 6, 949-973.
- Fishburn, P.C., 1983. A new characterization of simple majority. Economics Letters 13, 31-35.

References ii

- Llamazares, B., 2006. The forgotten decision rules: Majority rules based on difference of votes. Mathematical Social Sciences 51, 311-326.
- May, K., 1952. A set of independent, necessary and sufficient conditions for simple majority decision. Econometrica 20, 680-684.
- Miroiu, A., 2004. Characterizing majority rule: from profiles to societies. Economics Letters 85, 359-363.
- Ozkes, A.I., Sanver, M.R., 2017. Absolute qualified majoritarianism: How does the threshold matter? Economics Letters 153, 20-22.
- Quesada, A., 2010a. Monotonicity + efficiency + continuity = majority. Mathematical Social Sciences 60, 149-153.
- Quesada, A., 2010b. Two axioms for the majority rule. Economics Bulletin 30(4), 3033-3037.

- Quesada, A., 2013. The majority rule with arbitrators. Group Decision and Negotiation 22, 321-330.
- Woeginger, G.J., 2003. A new characterization of the majority rule. Economics Letters 81, 89-94.
- Xu, Y., Zhong, Z. (2010) Single profile of preferences with variable societies: a characterization of simple majority rule. Economics Letters 107, 119-121.
- Yi, J., 2005. A complete characterization of majority rules. Economics Letters 87, 109-112.